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Normal Systems

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Recently, P. Hall [*1*] established the existence of simple groups with non-trivial ascending composition systems in the sense of Kurosh [*2*]. In the following paper, alternate definitions of normal and composition systems will be offered. With these definitions, the following theorems will be proved.

THEOREM 1. *Let C be a chain of subgroups of a group G. Then*

(1) *C is a composition system of G if and only if C is a normal system of G with simple factors.*

(2) *If C is finite, then C is a normal (composition) system of G if and only if C is a normal (composition) series of G.*

(3) *If C is a finite composition system of G, then all composition systems of G are finite and isomorphic.*

THEOREM 2. *Every normal system of a group G can be refined to a composition system of G.*

A method for showing that the results (Theorems 3-5 in this paper) obtained for normal systems in the sense of Kurosh in [*3*, Part I, Section 3] still hold, will be indicated.

Let G be any group and C a set of subgroups of G. C is said to be a *chain of G* if it is linearly ordered by inclusion. A chain C is said to be *complete* if for each subset S of C, $\cup \{H \in S\}$ and $\cap \{H \in S\}$ are elements of C.

The chain C obtained from a chain C' by adding, for each subset S of C', $\cup \{H \in S\}$ and $\cap \{H \in S\}$ is called the *completion of C'*. If C is the completion of C', then C is complete and for $H \in C$, either $H = \cup \{M \in C' \mid H \supset M\}$ or $H = \cap \{M \in C' \mid H \subset M\}$.

Let C be any chain. If $M, N \in C$ are such that N is the immediate successor of M, then (M, N) is said to form a *jump in C*.

A complete chain C of G which contains $\{1\}$ and G is a *normal system in the sense of Kurosh* or a *K-normal system of G* if for every jump (M, N) in C,

$M \lhd N$. C is said to be a *normal system of* G if for $L, N \in C$, $L \subsetneq N$, there is a group $M \in C$ such that $L \subset M \subsetneq N$ and $M \lhd N$. Any normal system of G is also a K-normal system of G.

A normal system of G is a *composition system of* G if it cannot be refined.

If C is a normal system of G, then to every jump (M, N) in C, there corresponds the factor group N/M. N/M is said to be a *factor of* C. Two normal systems of G are said to be *isomorphic* if there exists a one-to-one correspondence between the factors of the two systems such that corresponding factors are isomorphic.

THEOREM 1. *Let C be a chain of G. Then*

(1) *C is a composition system of G if and only if C is a normal system of G with simple factors.*

(2) *If C is finite, then C is a normal (composition) system of G if and only if C is a normal (composition) series of G.*

(3) *If C is a finite composition system of G, then all composition systems of G are finite and are isomorphic.*

Proof of 1. Let D be a any normal system and (L, N) a jump in D. If N/L is not simple, then there is a group M such that $L \subsetneq M \subsetneq N$, $M \lhd N$. Then $D \cup \{M\}$ is a refinement of D. Therefore, if C is a composition system, the factors of C are simple.

Let C be a normal system of G with simple factors. If D is a normal system which refines C, then there is a group $M \in D$, $M \notin C$. If

$$L = \cup\{K \in C \mid K \subset M\} \qquad \text{and} \qquad N = \cap\{K \in C \mid K \supset M\}$$

then (L, N) is a jump in C with the property that $L \subsetneq M \subsetneq N$. Since D is a normal system, there is a group $M' \in D$ such that $L \subsetneq M \subset M' \subsetneq N$, $M' \lhd N$. This contradicts the fact that N/L is simple. Therefore C is a composition system.

Proof of 2. This follows immediately from the definitions and statement 1.

Proof of 3. Let n denote the length of C. Suppose that $D = \{H_b \mid b \in I\}$ with I infinite is a normal system of G. Let

$$D_1 = \{H_b \mid \{1\} \subsetneq H_b \subsetneq G, H_b \lhd G\}.$$

If D_1 has no largest element, then taking any n groups from D_1 together with $\{1\}$ and G yields a normal system of legnth $n + 1$. If D_1 has a largest element H_n, then, since I is infinite, we may continue this argument with

$$D_2 = \{H_b \mid \{1\} \subsetneq H_b \subsetneq H_n, \qquad H_b \lhd H_n\}.$$

If necessary, we continue further with $D_3, \cdots, D_m$, which can be defined similarly. We finally obtain a normal system $\{1\} \subsetneq H_1 \subsetneq \cdots \subsetneq H_n \subsetneq G$ of G, which is of length $n + 1$. It follows from statement 2 and Schreier's theorem, however, that this is impossible. Therefore every normal system of G is finite.

The result now follows from 2 and the Jordan-Holder theorem.

THEOREM 2. *Every normal system C of G can be refined to a composition system of G.*

Proof. By Zorn's Lemma, it suffices to show that every subset of $\{C' \mid C'$ is a normal system containing $C\}$ which is linearly ordered by inclusion has an upper bound. Let $K = \{C_\lambda \mid \lambda \in \Lambda\}$ be such a subset.

LEMMA. *$D' = \bigcup_{\lambda \in \Lambda} C_\lambda$ is a chain and D, the completion of D', is a K-normal system.*

Proof. See [2, p. 172].

LEMMA. *For $L, N \in D$, $L \subsetneq N$, there is a group $M \in D$ such that $L \subset M \subsetneq N$ and $M \lhd N$.*

Proof. Consider the possible cases.

Case 1. $L, N \in D'$.

Then there exist $C_\alpha, C_\beta \in K$ such that $L \in C_\alpha$, $N \in C_\beta$. Then

$$L, N \in C_\gamma = \max(C_\alpha, C_\beta).$$

Therefore there is a group $M \in C_\gamma \subset D$ with the required properties.

Case 2. $L \in D'$, $N = \cup \{H \in D' \mid N \supsetneq H\}$.

Then $N = \cup \{H \in D' \mid L \subsetneq H \subsetneq N\}$. Furthermore, it follows from Case 1 that for each such H, there is a group $H' \in D$ such that $H \subset H' \subsetneq N$, $H' \lhd N$. Then

$$L \subseteq M = \cap \{H' \mid H \in D',\ L \subsetneq H \subsetneq N\} \subsetneq \cup \{H \in D' \mid L \subsetneq H \subsetneq N\} = N$$

and $M \lhd N$. $M \in D$ since D is complete.

Case 3. $L \in D'$, $N = \cap \{H \in D' \mid N \subsetneq H\}$.

Let $H' = \cup \{H \in D' \mid N \supsetneq H\}$. If $H' = N$, the result follows from Case 2. If $H' \neq N$, then (H', N) is a jump in D. It then follows from the preceding lemma that $H' \lhd N$.

Case 4. $L = \cap \{H \in D' \mid L \subsetneq H\}$.

Since $L \subsetneq N$, there is a group $L' \in D'$ such that $L \subset L' \subsetneq N$. Then, by the preceding cases, there is a group $M \in D$ with $L' \subset M \subsetneq N$ and $M \lhd N$. M has the required properties.

Case 5. $L = \cup \{H \in D' \mid L \supsetneq H\}$.

For each $H \in D'$ with $L \supsetneq H$, there is a group $H' \in D$ such that $H \subset H' \subsetneq N$, $H' \lhd N$. If, for some such H', $L \subset H'$, then that group has the required properties. If there is no such H', then $\cup \{H' \mid H \in D', L \supsetneq H\} = L$. Then $L \lhd N$.

It follows from the above lemmas that D is a normal system. Therefore D is an upper bound for K.

COROLLARY. *Every group has a composition system.*

The following theorems can be proved by showing that the K-normal systems which Kurosh and Černikov [3] construct in order to prove the corresponding theorems for K-normal systems are indeed normal systems. This follows quite easily from the fact that the systems from which they are constructed are normal systems.

THEOREM 3. *If C is a normal system of G, then every subgroup of G has a normal system whose factors are isomorphic to subgroups of distinct factors of C.*

THEOREM 4. *Any two well-ordered ascending normal systems of an arbitrary group have isomorphic refinements which are well-ordered ascending normal systems.*

THEOREM 5. *If A is a well-ordered ascending normal system of G and B is an arbitrary normal system of G, then there is a refinement B' of B such that every factor of B' is isomorphic to a factor group of a subgroup of a factor of A.*

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